TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 308, Number 1, July 1988

TRAVELING WAVE SOLUTIONS OF A GRADIENT SYSTEM: SOLUTIONS WITH A PRESCRIBED WINDING NUMBER. II

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ABSTRACT. This paper completes the analysis begun in [**2**] concerning the existence of traveling wave solutions of a system of the form $u_t = u_{xx} + \nabla F(u)$, $u \in \mathbf{R}^2$. In [**2**] a notion of winding number for solutions was defined, and the proof that there exists a traveling wave solution with a prescribed winding number was reduced to a purely algebraic problem. In this paper the algebraic problem is solved.

1. Introduction.

A. *Statement of the problem.* This paper completes a study which we began in two previous papers [**1**, **2**].

We consider the reaction-diffusion system

$$u_{1t} = u_{1xx} + f_1(u_1, u_2), \quad u_{2t} = u_{2xx} + f_2(u_1, u_2) \tag{1A.1}$$

where u_1 and u_2 are functions of $(x, t) \in \mathbf{R} \times \mathbf{R}^+$. We assume that f_1 and f_2 are derived from some potential. That is, there exists a function $F \in C^2(\mathbf{R}^2)$ such that

$$f_i(u_1, u_2) = \frac{\partial F}{\partial u_i}(u_1, u_2), \qquad i = 1, 2, \tag{1A.2}$$

for each $u_1, u_2 \in \mathbf{R}$. By a traveling wave solution of (1A.1) we mean a nonconstant, bounded solution of the form

$$(u_1(x,t), u_2(x,t)) = (U_1(z), U_2(z)), \qquad z = x + \theta t.$$

A traveling wave solution corresponds to a solution which appears to be traveling with constant shape and velocity.

We wish to assume that F looks something like what is shown in Figure 1. Precise assumptions on F will be given shortly. For now we assume that F has at least three local maxima. These are at $(U_1, U_2) = A$, B and C where $F(A) < F(B) < F(C)$. We will be interested in traveling wave solutions which satisfy

$$\lim_{z \to -\infty} (U_1(z), U_2(z)) = A \quad \text{and} \quad \lim_{z \to +\infty} (U_1(z), U_2(z)) = B. \tag{1A.3}$$

Motivation for studying this problem is given in [**2**, §1E].

Note that if $(U_1(z), U_2(z))$ is a traveling wave solution and $(V_1(z), V_2(z)) = (U_1'(z), U_2'(z))$, then $(U_1(z), V_1(z), U_2(z), V_2(z))$ satisfies the system

$$\begin{aligned} U_1' &= V_1, \quad V_1' = \theta V_1 - F_{U_1}(U_1, U_2), \\ U_2' &= V_2, \quad V_2' = \theta V_2 - F_{U_2}(U_1, U_2). \end{aligned} \tag{1A.4}$$

Received by the editors January 16, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35B35, 35K35; Secondary 35B40.

Supported in part by the National Science Foundation under Grant #DMS 8401719.

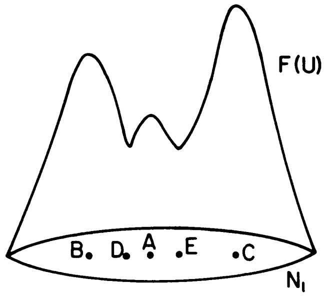


FIGURE 1

We are interested in solutions which satisfy

$$(1A.5) \qquad \lim_{z\to\infty}(U_1,U_2,V_1,V_2)=(A,\mathcal{O}) \quad \text{and} \quad \lim_{z\to+\infty}(U_1,U_2,V_1,V_2)=(B,\mathcal{O})$$

where $\mathcal{O}=(0,0)$.

In [**1**] it is proved that under certain assumptions on F, which are given shortly, there exists infinitely many traveling wave solutions of (1A.1) which satisfy (1A.3). That is, there exists infinitely many values of θ for which a solution of (1A.1), (1A.3) exists. We now wish to characterize the solutions (1A.4), (1A.5) by their nodal properties. We shall define a notion of winding number and prove that for each nonnegative integer K, there exists a solution of (1A.4), (1A.5) with winding number K. The proof of this result is split into two parts. In [**2**] we reduced the problem of finding a traveling wave solution with a prescribed winding number to a purely algebraic problem. We shall describe this algebraic problem shortly. In this paper we solve the algebraic problem.

B. *Assumptions on F*. The assumptions we make on F are those made in [**1**]. These are

(F1) $F\in C^2(\mathbf{R}^2)$.

(F2) F has at least three nondegenerate local maxima. These are at $U=A=(A_1,A_2)$, $B=(B_1,B_2)$, and $C=(C_1,C_2)$. F has at least two saddles. These are at $D=(D_1,D_2)$ and $E=(E_1,E_2)$.

(F3) $F(A)<F(B)<F(C)$ and $B_1<D_1<A_1<E_1<C_1$. Moreover, there exists an α_0 such that if α is any critical point of F with $\alpha\notin\{A,B,C\}$, then $F(\alpha)<F(A)-\alpha_0$. For convenience, we assume that $A=(0,0)$ and $F(A)=0$.

(F4) There exists W such that if $K<W$, then $\{U\colon F(U)\geq K\}$ is convex.

(F5) If $U_1=D_1$ or E_1, then $\partial F(U_1,U_2)/\partial U_1=0$ for all $U_2\in\mathbf{R}$.

(F6) Let $U=(U_1,U_2)$, $V=(V_1,V_2)$, and

$$(1B.1) \qquad \begin{aligned} &N_1=\{U\colon F(U)\geq W\}, \quad X_1=\{U\in N_1,U_1<D_1\},\\ &X_2=\{U\in N_1,D_1<U_1<E_1\}, \quad X_3=\{U\in N_1\colon E_1<U_1\}. \end{aligned}$$

Suppose that $(U(z),V(z))$ is a bounded solution of (1A.4) with $\theta=0$ which satisfies, for $i=1,2$, or 3,

(a) $U(z)\in X_i$ for all $z\in\mathbf{R}$,

(b) $F(U(z)) > F(A) - \alpha_0$ for some $z \in \mathbf{R}$
where α_0 was defined in (F3). Then $U(z)$ is identically equal to one of the critical points A, B, or C, and $V(z) = \mathcal{O}$ for all $z \in \mathbf{R}$.

Remarks concerning these assumptions are given in [**1**].

C. *The winding number.* Let

$$P_D = \{(U,V) \colon U_1 = D_1, V_1 = 0\}$$

and

$$P_E = \{(U,V) \colon U_1 = E_1, V_1 = 0\}.$$

Then P_D and P_E are two-dimensional subsets of the four-dimensional phase space. We wish to count the number of times solutions of (1A.4), (1A.5) wind around P_D and P_E. Perhaps the most important property of P_D and P_E is

PROPOSITION 1C.1. *P_D and P_E are invariant with respect to the flow given by* (1A.4). *That is, if* $(U_1(z_0), U_2(z_0), V_1(z_0), V_2(z_0)) \in P_D$ (P_E) *for some* z_0, *then* $(U_1(z), U_2(z), V_1(z), V_2(z)) \in P_D$ (P_E) *for all* $z \in \mathbf{R}$.

PROOF. From (1A.4) and (F5) we conclude that on P_D and P_E, $U_1' = V_1 = 0$ and $V_1' = \theta V_1 - F_{U_1}(U_1, U_2) = 0$. These two equalities prove the proposition.

An immediate consequence of this last result is

COROLLARY 1C.2. *If* $(U(z), V(z))$ *is a solution of* (1A.4), (1A.5) *then*

$$(U(z), V(z)) \notin P_D \cup P_E \quad \textit{for all } z.$$

It now makes sense to count the number of times a solution of (1A.4), (1A.5) winds around P_D and P_E. This is done as follows. Let

$$Q_D = \{(U,V) \colon U_1 = D_1, V_1 < 0 \text{ and } U \in N_1\},$$
$$Q_E = \{(U,V) \colon U_1 = E_1, V_1 > 0 \text{ and } U \in N_1\}.$$

DEFINITION. Suppose that $(U(z), V(z))$ is a solution of (1A.4), (1A.5). The winding number of U is defined as

$$h(U) = \operatorname{card}\{z \colon (U(z), V(z)) \in Q_D \cup Q_E\}. \tag{1C.1}$$

By card X we mean the cardinality of the set X. Remarks concerning this definition are given in [**1**].

D. *The main result.* Our main result is

THEOREM 1. *Let K be any positive integer. Then there exists a traveling wave solution $U(z)$ of* (1A.1), (1A.3) *such that either $h(U) = K$ or $h(U) = K + 1$.*

As we mentioned earlier this theorem is proved in two parts. In [**2**] we reduced the proof to a purely algebraic problem. In this paper we solve the algebraic problem, thus completing the proof of the theorem.

REMARK 1. The fact that we have either $h(U) = K$ or $h(U) = K + 1$ may be disturbing because we would expect there to exist a traveling solution such that $h(U) = K$. The reason that we obtain the weaker result is that we are counting the number of times a solution winds around two objects, namely P_D and P_E.

REMARK 2. We actually prove that for each positive integer K there exists at least two traveling wave solutions, each with winding number K or $K + 1$. The reason why this is true is explained in [**2**].

E. *The algebraic problem.* We now state the algebraic problem which we have (in [**2**]) reduced the proof of the theorem to.

Let F_4 be the set of words on the four elements $\{\alpha, \beta, \gamma, \delta\}$. That is, if $\Gamma \in F_4$, then we can express Γ as

$$\Gamma = \lambda_1^{e_1} \lambda_2^{e_2} \cdots \lambda_K^{e_K} \tag{1E.1}$$

where, for each i, $\lambda_i \in \{\alpha, \beta, \lambda, \delta\}$ and $e_i \in \{-1, 1\}$.

For $\Gamma \in F_4$, let Γ^* equal the subset of F_4 of all elements which upon cancellations equal Γ. For example if $\Gamma = \alpha\beta$, then $\alpha^2\beta\beta^{-1}\alpha^{-1}\beta \in \Gamma^*$.

If $\Gamma \in F_4$ is given by (1E.1), let

$$\omega(\Gamma) = \sum_{i=1}^{K} e_i \tag{1E.2}$$

and

$$h_1(\Gamma) = \sup_{1 \le j \le K} \sum_{i=1}^{j} e_i. \tag{1E.3}$$

REMARK. In [**2**] we used the notation $|\Gamma|$ instead of $h_1(\Gamma)$. To state the algebraic problem we must define another integer, $||\Gamma||$, for $\Gamma \in F_4$. We are not able to define this now, because it is necessary to develop quite a bit of the algebraic theory first. For now we assume that $||\Gamma||$ is well defined. It will be defined later.

We will now state the algebraic problem.

PROPOSITION 1E.1. *Let $\{\Gamma_k\}$, $k = 1, 2, \ldots$, be an infinite sequence of elements of F_4 which satisfy:*

(1E.4)
(a) $\Gamma_1 = \beta\gamma^{-1}$,
(b) *for each positive integer K there exists M such that if $k > M$, then $||\Gamma_k|| > K$,*
(c) *for each k there exists Γ_A, $\Gamma_B \in F_4$ and an integer r such that $\Gamma_A\Gamma_B \in \Gamma_k^*$ and $\Gamma_A(\alpha\beta\gamma^{-1}\delta^{-1})^r\Gamma_B \in \Gamma_{k+1}^*$.*

Let $h_k = \omega(\Gamma_A)$. Then for each positive integer K there exists k such that either $h_k = K$ or $h_k = K + 1$.

F. *Radial solutions of an elliptic system.* In a forthcoming paper [**3**] we consider radial solutions of the elliptic system

$$\begin{aligned} \Delta u_1 + f_1(u_1, u_2) &= 0, \\ \Delta u_2 + f_2(u_1, u_2) &= 0, \end{aligned} \tag{1F.1}$$

where u_1 and u_2 are functions of $x \in \mathbf{R}^n$, $n > 1$, and Δ is the usual Laplace operator. As in this paper we assume that $(f_1(U), f_2(U))$ satisfies (1A.2) for each $(u_1, u_2) \in \mathbf{R}^2$. By a radial solution of (1F.1) we mean a solution of the form $(u_1(x), u_2(x)) = (U_1(r), U_2(r))$, $r = ||x||$. Moreover, we assume that a radial solution satisfies $\lim_{r\to\infty}(U_1(r), U_2(r)) = (0, 0)$.

In a manner similar to what was done in this paper, we define a notion of winding number for solutions of (1F.1). We can then prove

THEOREM 2. *Assume that $n > 1$, and (f_1, f_2) satisfies* (1A.2) *where $F(U)$ satisfies* (F1)–(F6). *Moreover, assume that $A = (0,0)$. Then for each positive integer K there exists a radial solution of* (1F.1) *with winding number K or $K+1$.*

The proof of this theorem consists of two parts. We shall first reduce the proof of Theorem 2 to the algebraic problem, Proposition 1E.1, of this paper. Hence, the proof of Proposition 1E.1, given here, will imply the validity of Theorem 2.

G. *A preliminary result.* Suppose that $\{\Gamma_k\}$ satisfies (a), (b), and (c) of Proposition 1E.1, and let $h_k = \omega(\Gamma_A^*)$ be as in Proposition 1E.1. In this section we prove

PROPOSITION 1G.1. *Let k be a positive integer. Then there exists $\Gamma_A, \Gamma_B, H \in F_4$ and an integer r such that*

$$\text{(1G.1)} \qquad \begin{array}{ll} \text{(a)} & \Gamma_k = \Gamma_A \Gamma_B, \\ \text{(b)} & \Gamma_A H (\alpha\beta\gamma^{-1}\delta^{-1})^r H^{-1} \Gamma_B \in \Gamma_{k+1}^*. \end{array}$$

Moreover, $h_k = \omega(\Gamma_A H)$.

REMARK. This proposition relates Γ_k and Γ_{k+1}^*, while Proposition 1E.1 relates Γ_k^* and Γ_{k+1}^*.

PROOF. From the assumptions of Proposition 1E.1 there exists $\Gamma'_A, \Gamma'_B \in F_4$ and an integer r such that $\Gamma'_A \Gamma'_B \in \Gamma_k^*$ and $\Gamma'_A (\alpha\beta\gamma^{-1}\delta^{-1})^r \Gamma'_B \in \Gamma_{k+1}^*$. Let $\hat{\Gamma}_A$ equal Γ'_A with all possible cancellations and $\hat{\Gamma}_B$ equal Γ'_B with all possible cancellations. Then clearly $\hat{\Gamma}_A \hat{\Gamma}_B \in \Gamma_k^*$ and $\hat{\Gamma}_A (\alpha\beta\gamma^{-1}\delta^{-1})^r \hat{\Gamma}_B \in \Gamma_{k+1}^*$. Moreover, $h_k = \omega(\Gamma'_A) = \omega(\hat{\Gamma}_A)$.

Let H equal the maximal element of F_4 so that we can write $\hat{\Gamma}_A = \Gamma_A H$ and $\hat{\Gamma}_B = H^{-1}\Gamma_B$ for some $\Gamma_A, \Gamma_B \in F_4$. Then $\Gamma_A \Gamma_B \in \Gamma_k^*$. Since there are no cancellations in $\Gamma_A \Gamma_B$ we have that $\Gamma_k = \Gamma_A \Gamma_B$. Moreover,

$$\Gamma_A H (\alpha\beta\gamma^{-1}\delta^{-1})^r H^{-1} \Gamma_B = \hat{\Gamma}_A (\alpha\beta\gamma^{-1}\delta^{-1})^r \hat{\Gamma}_B \in \Gamma_{k+1}^*,$$

and

$$\omega(\Gamma_A H) = \omega(\hat{\Gamma}_A) = h_k.$$

This is what we wished to prove.

H. *Remarks concerning the proof.* The major difficulty in the proof of Theorem 1 is that the formulas given in (1G.1) are quite complicated and difficult to work with. For one thing, H can be an arbitrary element of F_4. Moreover, these formulas tell us how to compute Γ_{k+1}^*, not Γ_k, from Γ_k. To obtain Γ_{k+1} we must make cancellations. After each cancellation, information is lost, unless we are careful to do quite a bit of bookkeeping. To each Γ_k we assign an algebraic structure, which we call an A^*-decomposition. This will allow us to keep track of the essential information abut the Γ_k's as we increase k and then make cancellations. Of course, some information about the Γ_k's will be important, while other information will just get in the way. In (1G.1b), the term $(\alpha\beta\gamma^{-1}\delta^{-1})^r$ will be very important, while the terms H and H^{-1} will be troublesome. The A^*-decomposition will be defined in such a way as to keep track of just the useful information.

The definition of an A^*-decomposition is given in §2. In §3, we construct an A^*-decomposition for each Γ_k. An important preliminary result is proved in §4. The proof of Theorem 1 is completed in §5.

2. Notation and definitions. Assume that $\Gamma \in F_4$ is given by

$$\Gamma = \lambda_1^{e_1}\lambda_2^{e_2}\cdots\lambda_K^{e_K} \tag{2.1}$$

where each $\lambda_i \in \{\alpha, \beta, \gamma, \delta\}$ and $e_i \in \{-1, +1\}$. For each positive integer K, let $Y_K = \{1, 2, \ldots, K\}$. For Γ given by (2.1) let $Y_\Gamma = Y_K$.

DEFINITION 2.1. Assume that $\Gamma \in F_4$ is given by (2.1). By an *A-decomposition* of Γ we mean a four-tuple (Z, G, H, Φ) such that

(a) Z, G, and H are disjoint subsets of Y_Γ,

(b) $Y_\Gamma = Z \cup G \cup H$,

(c) Φ is a bijection from G onto H such that for all $g, g_1 \in G$,

  (i) $g < \Phi(g)$,

  (ii) $\lambda_g = \lambda_{\Phi(g)}$,

  (iii) $e_g = -e_{\Phi(g)}$.

REMARK. It is possible that $G = H = \Phi$.

For $\eta \in Y_\Gamma$, $g \in G$ we write

$$\eta \subset g \quad \text{if } g < \eta < \Phi(g). \tag{2.2}$$

If (2.2) is not true we write $\eta \not\subset g$. For $\eta \in Y_\Gamma$, let

$$G_\eta = \{g \in Y_\Gamma \colon \eta \subset g\}.$$

If (Z, G, H, Φ) is an A-decomposition of Γ, we partition Z into equivalence classes as follows. If $z_1, z_2 \in Z$, then $z_1 \sim z_2$ if and only if $G_{z_1} = G_{z_2}$. For $\eta \in Z$, let $Z_\eta = \{z \in Z \colon z \sim \eta\}$. If (Z, G, H, Φ) is an A-decomposition of Γ, we write

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_J \tag{2.3}$$

where the Z_i are the distinct equivalence classes of Z.

Let F_2 be the set of words generated by the elements a and b. Let $\psi\colon F_4 \to F_2$ be the homomorphism generated by

$$\psi(\alpha) = \psi(\delta) = a \quad \text{and} \quad \psi(\beta) = \psi(\gamma) = b. \tag{2.4}$$

Suppose that $\Gamma \in F_4$ is given by (2.1) and $Y \subset Y_\Gamma$ is given by $Y = \{\eta_1, \eta_2, \ldots, \eta_J\}$. Let

$$\psi(Y) = \psi[\lambda_{\eta_1}^{e_{\eta_1}} \cdots \lambda_{\eta_J}^{e_{\eta_J}}] \quad \text{and} \quad \omega(Y) = \sum_{i=1}^{J} e_{\eta_i}.$$

Finally, let I be the identity element in F_4.

DEFINITION 2.2. Let (Z, G, H, Φ) be an A-decomposition of Γ. We say that (Z, G, H, Φ) is an A^*-decomposition if $\psi(Z_i) = I$ for each equivalence class Z_i.

In the next section we prove that for each k, there exists an A^*-decomposition of Γ_k, where the Γ_k are as in Proposition 1E.1. We must first introduce some more notation.

Let Γ be as in (2.1) and let (Z, G, H, Φ) be an A-decomposition of Γ. Let

$$\begin{aligned} &\text{(a)} \quad |\Gamma| = K, \text{ and} \\ &\text{(b)} \quad Z_0 = \{z \in Z \colon G_z = \varnothing\}. \end{aligned} \tag{2.5}$$

That is, $|\Gamma|$ is the number of elements in Γ. If $Z_0 = \{\delta_1, \delta_2, \ldots, \delta_J\}$, let

$$\Gamma_0 = \lambda_{\delta_1}^{e_{\delta_1}} \cdots \lambda_{\delta_J}^{e_{\delta_J}}. \tag{2.6}$$

Finally, let

$$||\Gamma|| = h_1(\Gamma_0), \tag{2.7}$$

where $h_1(\Gamma)$ was defined in (1E.3).

We have now defined $||\Gamma||$ which is needed in the statement of Proposition 1E.1. In the next section we shall prove that for each k, there exists an A^*-decomposition of Γ.

REMARK. Each Γ may have more than one A^*-decomposition. Then $||\Gamma||$ will depend on the particular A^*-decomposition. We assume that $||\Gamma||$ is defined relative to the specific A^*-decomposition explicitly constructed in the next section of this paper.

3. An A^*-decomposition of Γ_k. Let $\{\Gamma_k\}$ be as in Proposition 1E.1. In this section we prove

PROPOSITION 3.1. *For each k there exists an A^*-decomposition of Γ_k.*

The A^*-decomposition of Γ_k will be denoted by (Z_k, G_k, H_k, Φ_k). We also set $Y^k = Y_{\Gamma_k}$. The A^*-decompositions are defined inductively. Recall that $\Gamma_1 = \beta\gamma^{-1}$. Hence, $Y^1 = \{1, 2\}$. Let $Z_1 = \{1, 2\}$, $G_1 = \varnothing$, and $H_1 = \varnothing$. Since $G_1 = \varnothing$, it is not necessary to define Φ_1. One easily checks that this defines an A^*-decomposition of Γ_1.

Suppose that there exists an A^*-decomposition, $(A_{k-1}, G_{k-1}, H_{k-1}, \Phi_{k-1})$, of Γ_{k-1}. We wish to define an A^*-decomposition of Γ_k.

Recall the basic formulas derived in Proposition 1G.1. There exists $\Gamma_A, \Gamma_B, H \in F_4$ and an integer r such that

$$\Gamma_{k-1} = \Gamma_A \Gamma_B \tag{3.1a}$$

and

$$\Gamma'_k \equiv \Gamma_A H(\alpha\beta\gamma^{-1}\delta^{-1})^r H^{-1} \Gamma_B \in \Gamma^*_k. \tag{3.1b}$$

The primary difficulty with this formula is that it is for Γ'_k and not Γ_k. That is, there may be cancellations. We first define an A^*-decomposition for Γ'_k, and then show how to handle each cancellation. The A^*-decomposition for Γ'_k will be denoted by $\{Z', G', H', \Phi'\}$.

Let

$$N_A = |\Gamma_A|, \quad N_B = |\Gamma_B|, \quad N_H = |H|, \quad \text{and} \quad r_1 = 4|r|, \tag{3.2}$$

where if Γ is given by (2.1), then $|\Gamma|$ is given by (2.5a). Then

$$|Y^{k-1}| = N_A + N_B \quad \text{and} \quad |Y_{\Gamma'_k}| = N_A + N_B + 2N_H + r_1 \equiv r'.$$

We now define Z', G', and H'.

(I) If $j \le N_A$, then

$$j \in \begin{cases} Z', \\ G', \\ H', \end{cases} \quad \text{if and only if} \quad j \in \begin{cases} Z_{k-1}, \\ G_{k-1}, \\ H_{k-1}. \end{cases}$$

(II) If $N_A < j \le N_A + N_H$, then $j \in G'$.
(III) If $N_A + N_H < j \le N_A + H_H + r_1$, then $j \in Z'$.
(IV) If $N_A + N_H + r_1 < j \le N_A + 2N_H + r_1$, then $j \in H'$.
(V) If $N_A + 2N_H + r_1 < j \le r'$, then

$$j \in \begin{cases} Z', \\ G', \\ H', \end{cases} \quad \text{if and only if} \quad j - (2N_H + r_1) \in \begin{cases} Z_{k-1}, \\ G_{k-1}, \\ H_{k-1}. \end{cases}$$

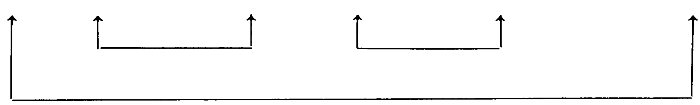
We now define Φ'. Assume that $g \in G'$.
(I) If $g \le N_A$ and $\Phi_{k-1}(g) \le N_A$, let $\Phi'(g) = \Phi_{k-1}(g)$.
(II) If $g \le N_A$ and $\Phi_{k-1}(g) > N_A$, let $\Phi'(g) = \Phi_{k-1}(g) + 2N_H + r_1$.
(III) If $N_A < g \le N_A + N_H$, let $\Phi'(g) = 2N_A + 2N_H + r_1 - g$.
(IV) If $N_A + 2N_H + r_1 < g \le r'$, let $\Phi'(g) = \Phi_{k-1}(g) + 2N_H + r_1$.

The idea behind what we just did is that the integers corresponding to elements of Γ_A and Γ_B inherit their status, that is whether they belong to Z', G', or H' from their status in $(Z_{k-1}, G_{k-1}, H_{k-1}, \Phi_{k-1})$. The integers corresponding to elements in H, $(\alpha\beta\gamma^{-1}\delta^{-1})^r$, and H^{-1} become elements of G', Z', and H', respectively.

To prove that (Z', G', H', Φ') is an A-decomposition of Γ_k' we must show that the conditions of Definition 2.1 are satisfied. Certainly (a) and (b) of Definition 2.1 are satisfied. In words, the reason that (c) is satisfied is because if one considers (3.1), then the integers in Y' corresponding to elements in Γ_A and Γ_B inherit their "status" from $(Z_{k-1}, G_{k-1}, H_{k-1}, \Phi_{k-1})$ which is assumed to be an A-decomposition. Moreover, Φ' preserves the natural one-to-one correspondence between the integers in Y' corresponding to H and the integers in Y' corresponding to elements in H^{-1}. Instead of writing down a detailed proof, which would involve even more notation, we present some examples which will illustrate that the above construction is a natural one.

Suppose that

$$\begin{array}{cccccccccccccccccc} & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ (3.3) & \Gamma = & \alpha & \beta & \alpha & \gamma & \beta & \gamma^{-1} & \gamma^{-1} & \beta & \alpha & \alpha & \delta^{-1} & \alpha^{-1} & \gamma^{-1} & \delta^{-1} & \beta^{-1} & \delta^{-1} \end{array}$$



Then $|Y_\Gamma| = 16$. Let

$$Z = \{1, 3, 5, 6, 8, 10, 11, 13, 14, 16\},$$

$$G = \{2, 4, 9\}, \quad \text{and} \quad H = \{7, 12, 15\}.$$

Define Φ by

$$\Phi(2) = 15, \quad \Phi(4) = 7, \quad \text{and} \quad \Phi(9) = 12.$$

One easily checks that $\{Z, G, H, \Phi\}$ is an A-decomposition of Γ. Let

$$Z^1 = \{1, 16\}, \quad Z^2 = \{3, 8, 13, 14\}, \quad Z^3 = \{5, 6\}, \quad \text{and} \quad Z^4 = \{10, 11\}.$$

Then $\{Z^1, Z^2, Z^3, Z^4\}$ gives the partition of Z. Moreover,

$$\psi(Z^1) = aa^{-1} = I, \qquad \psi(Z^2) = abb^{-1}a^{-1} = I,$$
$$\psi(Z^3) = bb^{-1} = I, \qquad \psi(Z^4) = aa^{-1} = I.$$

Since $\psi(Z^j) = I$ for each j, $\{Z, G, H, \Phi\}$ defines an A^*-decomposition of Γ.

We now give an example to illustrate how the A^*-decomposition changes as we go from Γ_{k-1} to Γ'_k. Suppose that

$$\begin{array}{rcccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \Gamma_{k-1} = & \beta & \alpha & \beta & \gamma^{-1} & \alpha^{-1} & \beta^{-1} \end{array}$$

Then $|Y_{k-1}| = 6$. Let

$$Z_{k-1} = \{1, 3, 4, 6\}, \quad G_{k-1} = \{2\}, \quad H_{k-1} = \{5\} \quad \text{and} \quad \Phi(2) = 5.$$

One easily verifies that this defines an A^*-decomposition of Γ_{k-1}. Let

$$\Gamma_A = \beta\alpha\beta, \quad \Gamma_B = \gamma^{-1}\alpha^{-1}\beta^{-1}, \quad H = \alpha\beta, \quad \text{and} \quad r = -1.$$

Then

$$\Gamma_{k-1} = \Gamma_A \Gamma_B,$$

and

$$\Gamma'_k = \Gamma_A H(\alpha\beta\gamma^{-1}\delta^{-1})^{-1} H \Gamma_B$$

$$\begin{array}{rcccccccccccccc} & 1 & 2 & 3 & & & & & & & & 4 & 5 & 6 \\ = & \beta & \alpha & \beta & \alpha & \beta & \delta & \gamma & \beta^{-1} & \alpha^{-1} & \beta^{-1} & \alpha^{-1} & \gamma^{-1} & \alpha^{-1} & \beta^{-1} \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{array} \tag{3.4}$$

The lower integers in (3.4) indicate the integer in Y'_k which the element in Γ'_k corresponds to. The upper integers indicate the integer in Y_{k-1} which the particular element belongs to. An A^*-decomposition of Γ'_k is

$$Z^1 = \{1, 3, 6, 7, 8, 9, 12, 14\},$$
$$G^1 = \{2, 4, 5\}, \qquad H^1 = \{10, 11, 13\},$$
$$\Phi(2) = 13, \qquad \Phi(4) = 11, \qquad \Phi(5) = 10.$$

We now return to the general situation. We claim that the A-decomposition of Γ'_k is actually an A^*-decomposition. Because Γ_{k-1} is an A^*-decomposition and $\psi((\alpha\beta\gamma^{-1}\delta^{-1})^r) = I$, this is obvious.

This takes care of Γ'_k. We must now discuss what happens when there are cancellations. Note that Γ_k is obtained from Γ'_k after a finite number of cancellations. We show that everything is fine after one cancellation. To obtain the desired result we just repeat the same argument a finite number of times.

Assume that Γ is given by

$$\Gamma = \lambda_1^{e_1} \lambda_2^{e_2} \cdots \lambda_J^{e_J}$$

and, for some $j < J$, $\lambda_j = \lambda_{j+1}$ and $e_j = -e_{j+1}$. That is, $\lambda_j^{e_j}$ cancels with $\lambda_{j+1}^{e_{j+1}}$ in Γ. Let

$$\Gamma' = \lambda_1^{e_1} \cdots \lambda_{j-1}^{e_{j-1}} \lambda_{j+2}^{e_{j+2}} \cdots \lambda_J^{e_J} = \eta_1^{f_1} \cdots \eta_{J-2}^{f_{J-2}}.$$

Hence, Γ' is Γ after the cancellation. Define $\varsigma\colon Y_{\Gamma'} \to Y_\Gamma$ by

$$\varsigma(i) = \begin{cases} i & \text{if } 1 \le i \le j-1, \\ i+2 & \text{if } j+2 \le i \le J-2. \end{cases} \tag{3.5}$$

Then

$$\eta_i = \lambda_{\varsigma(i)} \quad \text{and} \quad f_i = e_{\varsigma(i)}.$$

Let $\{Z, G, H, \Phi\}$ be an A^*-decomposition of Γ. We prove that there is a "natural" A^*-decomposition of Γ' which we will denote by $\{Z', G', H', \Phi'\}$. There are a number of cases to consider. We present, in detail, only a few of them.

(I) Suppose that $j \in Z$ and $j+1 \in Z$.

For $1 \le i \le K-2$, let

$$i \in \begin{cases} Z', \\ G', \\ H', \end{cases} \quad \text{if and only if} \quad \varsigma(i) \in \begin{cases} Z, \\ G, \\ H. \end{cases} \tag{3.6}$$

To define Φ', assume that $g \in G'$. Let $\Phi'(g) = \varsigma(\Phi(g))$. That is, $\{Z', G', H', \Phi'\}$ inherit their properties from $\{Z, G, H, \Phi\}$. Since $\{Z, G, H, \Phi\}$ is an A-decomposition, so is $\{Z', G', H', \Phi'\}$. To see why $\{Z', G', H', \Phi'\}$ is an A^*-decomposition, suppose that the partition of Z is given by

$$Z = Z^1 \cup Z^2 \cup \cdots \cup Z^J. \tag{3.7}$$

Then there exists k such that $\{j, j+1\} \subset Z^k$. For convenience we assume that $k = 1$. Let $Z_1' = Z^1 \backslash \{j, j+1\}$. Since $\psi(Z^1) = I$ and $\psi(\lambda_j^{e_j} \lambda_{j+1}^{e_{j+1}}) = I$, it is clear that $\psi(Z_1') = I$. It is not hard to see then that $\{Z', G', H', \Phi'\}$ is an A^*-decomposition of Γ'.

(II) Suppose that $j \in Z$ and $j+1 \in G$.

We define $\{Z', G', H', \Phi'\}$ almost as before. The difference is that now $\Phi(j+1)$ has lost its "partner", $j+1$, due to cancellation. Hence, after renumbering, we assign $\Phi(j+1)$ to belong to Z'. Here are some of the details.

For $1 \le i \le k-2$, $\varsigma(i) \ne \Phi(j+1)$, let

$$i \in \begin{cases} Z', \\ G', \\ H', \end{cases} \quad \text{if and only if} \quad \varsigma(i) \in \begin{cases} Z, \\ G, \\ H. \end{cases} \tag{3.8}$$

If $\varsigma(i) = \Phi(j+1)$, let $i \in Z'$. To define Φ', assume that $g \in G'$. Let $\Phi'(g) = \varsigma(\Phi(g))$.

Assume that the partition of Z is given by (3.7). For convenience we assume that $j \in Z^1$. In Case I, the partition of Z' was easily obtained from the partition of Z. Now we must be more careful.

Recall the notation introduced in (2.2). Let

$$M = \{z \in Z\colon z \subset j+1, \text{ and } z \not\subset g \text{ if } g \in G \text{ and } g \subset j+1\}.$$

We consider two cases; these are if $M = \varnothing$ or $M \neq \varnothing$. First assume that $M = \varnothing$. Suppose that for $1 < i \leq J$,

$$Z^i = \{z_1^i, \ldots, z_I^i\}. \tag{3.9a}$$

Let

$$\hat{Z}^i = \{\varsigma^{-1}(z_1^i), \ldots, \varsigma^{-1}(z_I^i)\}. \tag{3.9b}$$

Since, for $1 < i \leq J$, Z^i and $\hat{Z}^i$ really correspond to the same element of F_4, we have that, for $1 < i \leq J$,

$$\psi(\hat{Z}^i) = \psi(Z^i) = \mathcal{I}. \tag{3.9c}$$

It remains to consider the case $i = 1$. Suppose that $Z^1 = \{z_1, \ldots, z_I\}$, and choose l so that $z_l = j$. Let

$$Z_0^1 = \{\varsigma^{-1}(z_1), \ldots, \varsigma^{-1}(z_{l-1}), \varsigma^{-1}(\Phi(j+1)), \varsigma^{-1}(z_{l+1}), \ldots, \varsigma^{-1}(z_I)\}. \tag{3.10}$$

One easily shows that the partition of Z' is $Z' = Z_0^1 \cup \hat{Z}^2 \cup \cdots \cup \hat{Z}^I$. We must still show that $\psi(Z_0^1) = \mathcal{I}$.

Let F^1 and F^2 be the elements of F_4 corresponding to Z^1 and Z_0^1, respectively. The only difference between F^1 and F^2 is that $\lambda_j^{e_j}$ in F^1 is replaced by $\lambda_k^{e_k}$, where $k = \Phi(j+1)$, in F^2. By assumption

$$\lambda_j^{e_j} = [\lambda_{j+1}^{e_{j+1}}]^{-1} = \lambda_k^{e_k}. \tag{3.11}$$

Hence, $F^1 = F^2$. Since $\psi(Z^1) = \mathcal{I}$, it follows that $\psi(Z_0^1) = \mathcal{I}$.

It remains to consider the case $M \neq \varnothing$. Then there exists $k \neq 1$ such that $M = Z^k$. For convenience we assume that $k = 2$. Let

$$Z_A = \{z \in Z^1 : z \leq j\} \quad \text{and} \quad Z_B = \{z \in Z^1 : z > j\}.$$

Of course, $Z^1 = Z_A Z_B$. Suppose that $Z_A = \{z_1, \ldots, z_I\}$. Note that $z_I = j$. Let

$$Z_A' = \{z_1, \ldots, z_{I-1}, \Phi(j+1)\}.$$

Using (3.11) we find that

$$\psi(Z^1) = \psi(Z_A Z_B) = \psi(Z_A' Z_B) = \psi(\hat{Z}_A' \hat{Z}_B) \tag{3.12}$$

where $\hat{Z}_A'$ and $\hat{Z}_B$ are defined as in (3.9b). For $2 \leq i \leq J$, let $\hat{Z}^i$ be defined as in (3.9). The partition of Z' is then

$$Z' = (\hat{Z}_A' \hat{Z}^2 \hat{Z}_B) \cup \hat{Z}^3 \cup \cdots \cup \hat{Z}^J.$$

As in (3.9c) we have that if $3 \leq i \leq J$,

$$\psi(\hat{Z}^i) = \psi(Z^i) = \mathcal{I}.$$

Moreover, using (3.11),

$$\begin{aligned} \psi(\hat{Z}_A' \hat{Z}^2 \hat{Z}_B) &= \psi(Z_A' Z^2 Z_B) = \psi(Z_A')\psi(Z^2)\psi(Z_B) \\ &= \psi(Z_A')\mathcal{I}\psi(Z_B) = \psi(Z_A' Z_B) = \psi(Z^1) = \mathcal{I}. \end{aligned}$$

Hence $\{Z', G', H', \Phi'\}$ is an A^*-decomposition.

There are still four more cases to consider for the proof of Proposition 3.1. These are

(III) $j \in Z$, $j+1 \in H$,
(IV) $j \in G$, $j+1 \in Z$,
(V) $j \in H$, $j+1 \in Z$,
(VI) $j \in G$, $j+1 \in H$.

Since the analysis for each of these cases is not much different from case II, we only state the definition of $\{Z', G', H', \Phi'\}$ in each case.

For case III, let $k = \Phi^{-1}(j+1)$. Let ς be the map defined in (3.5). For $1 \le i \le k-2$, $\varsigma(i) \neq k$, assume that (3.8) holds. If $\varsigma(i) = k$, assume that $i \in Z'$. To define Φ, let $g \in G'$. Then let

$$\Phi'(g) = \varsigma(\Phi(g)). \tag{3.13}$$

Next consider case IV. For $1 \le i \le k-2$, $\varsigma(i) \neq \Phi(j)$ assume that (3.8) holds. If $\varsigma(i) = \Phi(j)$, assume that $i \in Z'$. Define Φ' by (3.13).

Next consider case V. Let $k = \Phi^{-1}(j)$. For $1 \le i \le k-2$, $\varsigma(i) \neq k$, assume that (3.8) holds. If $\varsigma(i) = k$, assume that $i \in Z'$. Define Φ' by (3.13).

Finally, consider case VI. We must then have $\Phi(j) = j+1$. The A^*-decomposition is given by (3.8) and (3.13).

4. A preliminary result. We begin with some definitions. For each k, let $\{Z_k, G_k, H_k, \Phi_k\}$ be the A^*-decomposition of Γ_k. For $\eta \in \Gamma_k$, let G_η be as defined following (2.2). If $G_\eta \neq \emptyset$, let

$$g_\eta = \sup\{g \colon g \in G_\eta\}. \tag{4.1}$$

If $\Gamma_k = \lambda_1^{e_1} \cdots \lambda_J^{e_J}$, then, for $1 \le j \le J$, let

$$\omega_k(j) = \sum_{i=1}^{j} e_i.$$

If the meaning is clear, we write $\omega(j)$ instead of $\omega_k(j)$. In this section we prove

PROPOSITION 4.1. *Fix $K > 0$. Suppose that Γ_{k-1} satisfies*

(a) $\|\Gamma_{k-1}\| \le K+1$,

(b) *for each $z \in Z_{k-1}$, if $G_z \neq \emptyset$, then $\omega_{k-1}(z) \le K+1$ if and only if $\omega_{k-1}(g_z) \le K+1$.*

Then either $h_k = K$ or $K+1$, or Γ_k satisfies (a) *and* (b).

REMARK. This proposition implies the proof of Proposition 1E.1 because (a) is not satisfied for all Γ_k. We have not yet verified, however, (b) of Proposition 1E.1. This will be done in the next section.

PROOF. From Proposition 1G.1 we may choose Γ_A, Γ_B, $H \in F_4$ and r such that

$$\begin{aligned} &\text{(a)} \quad \Gamma_{k-1} = \Gamma_A \Gamma_B, \\ &\text{(b)} \quad \Gamma_k' = \Gamma_A H (\alpha\beta\gamma^{-1}\delta^{-1})^r H^{-1} \Gamma_B \in \Gamma_k^*, \\ &\text{(c)} \quad h_k = \omega(\Gamma_A H). \end{aligned} \tag{4.2}$$

The main difficulty is that there may be cancellations in Γ_k'. We may assume that no part of Γ_A cancels with a part of H, and no part of H^{-1} cancels with a part of

Γ_B. This is for the following reason. Suppose that a part of Γ_A did cancel with a part of H. That is,

$$\Gamma_A = \Gamma_A^1 H_1 \quad \text{and} \quad H = H_1^{-1} H_2.$$

Let

$$\mathcal{X} = (\alpha\beta\gamma^{-1}\delta^{-1})^r. \tag{4.3}$$

Then

$$\Gamma_k' = \Gamma_A^1 H_1 H_1^{-1} H_2 \mathcal{X} H_2^{-1} H_1 \Gamma_B.$$

Hence

$$\Gamma_k' = \Gamma_A^1 H_2 \mathcal{X} H_2^{-1} \Gamma_B^1 \tag{4.4b}$$

where $\Gamma_B^1 = H_1\Gamma_B$. Note that

$$\Gamma_{k-1} = \Gamma_A\Gamma_B = \Gamma_A^1 H_1 \Gamma_B = \Gamma_A^1\Gamma_B^1. \tag{4.4a}$$

Moreover,

$$h_k = \omega(\Gamma_A H) = \omega(\Gamma_A^1 H_1 H_1^{-1} H_2) = \omega(\Gamma_A^1 H_2). \tag{4.4c}$$

Then (4.4) is of the same form as (4.2), the only difference is that Γ_A is replaced with Γ_A^1 and H is replaced by H_2. A similar analysis holds if part of H^{-1} cancels with part of Γ_B.

Now suppose that part of H cancels with part of $\mathcal{X}$, say $H = H_1\mathcal{X}_1^{-1}$ and $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$. Then, from (4.2),

$$\Gamma_k' = \Gamma_A H_1 \mathcal{X}_1^{-1}\mathcal{X}_1\mathcal{X}_2\mathcal{X}_1 H_1^{-1}\Gamma_B = \Gamma_A H_1 \mathcal{X}_2\mathcal{X}_1 H_1^{-1}\Gamma_B, \tag{4.5a}$$

and

$$h_k = \omega(\Gamma_A H) = \omega(\Gamma_A H_1 \mathcal{X}_1^{-1}) = \omega(\Gamma_A H_1) - \omega(\mathcal{X}_1).$$

Because $\omega(\mathcal{X}_1\mathcal{X}_2) = \omega(\mathcal{X}_1) + \omega(\mathcal{X}_2) = 0$, we conclude that

$$h_k = \omega(\Gamma_A H_1 \mathcal{X}_2). \tag{4.5b}$$

A similar analysis holds if part of $\mathcal{X}$ cancels with part of H^{-1}. Hence we conclude that there exists $\Gamma_A, H, \mathcal{X}_1, \mathcal{X}_2, \Gamma_B \in F_4$ and an integer r such that

$$\begin{aligned} &\text{(a)} \quad \Gamma_k' = \Gamma_A H \mathcal{X}_2\mathcal{X}_1 H^{-1}\Gamma_B \in \Gamma_k^*, \\ &\text{(b)} \quad h_k = \omega(\Gamma_A H \mathcal{X}_2), \\ &\text{(c)} \quad \mathcal{X} = \mathcal{X}_1\mathcal{X}_2 = (\alpha\beta\gamma^{-1}\delta^{-1})^r, \end{aligned} \tag{4.6}$$

and if $H \neq \emptyset$, then *no more cancellations take place.*

Let $N_A = |\Gamma_A|$, $N_B = |\Gamma_B|$, $N_H = |H|$, and $r_1 = 4|r|$, where $|\Gamma|$ was defined in (2.5a). Suppose that $1 \le y \le |\Gamma_k'|$. We say that

$$y \in \begin{cases} \Gamma_A, \\ H, \\ \mathcal{X}, \\ H^{-1}, \\ \Gamma_B, \end{cases} \quad \text{if} \quad \begin{cases} 1 \le y \le N_A, \\ N_A < y \le N_A + N_H, \\ N_A + N_H < y \le N_A + N_H + r_1, \\ N_A + N_H + r_1 < y \le N_A + 2N_H + r_1, \\ N_A + 2N_H + r_1 < y \le N_A + 2N_H + r_1 + N_B. \end{cases}$$

The next result follows from (4.6).

LEMMA 4.2. *If $y \in \mathcal{X}$, then $h_k \le \omega(y) \le h_k + 2$.*

LEMMA 4.3. *Assume that $h_k \ne K$ or $K+1$. Then $\omega(N_A + N_H) \le K+1$ if and only if $\omega(y) \le K+1$ for all $y \in \mathcal{X}$.*

PROOF. We consider a number of cases, which we present in outline form. Let $y_i = N_A + N_H + i$ for $i = 0, 1, 2, 3 \dots$.

(A) Assume that $\omega(y_0) \le K+1$.

(i) Assume that $e_{y_1} > 0$.

(a) Assume that $\lambda_{y_1} = \alpha$. Then $\mathcal{X}_1 = \alpha\beta\gamma^{-1}\delta^{-1}\dots$, and $\omega(y_0) = h_k$. Since $\omega(y_0) \le K+1$, and $h_k \ne K$ or $K+1$, we conclude that $\omega(y_0) \le K-1$. From (4.7) we conclude that $\omega(y) \le \omega(y_0) + 2 \le K+1$ for $y \in \mathcal{X}$.

The case $\lambda_{y_1} = \delta$ is similar to this one so we do not include the proof.

(b) Assume that $\lambda_{y_1} = \beta$. Then $\mathcal{X}_2\mathcal{X}_1 = \beta\gamma^{-1}\delta^{-1}\alpha\dots$, and $h_k = \omega(y_3) = \omega(y_0) - 1 \le K$. Therefore, $h_k \le K-1$ and $\omega(y_0) \le h_k + 1 \le K$. Since $h_1(\mathcal{X}_2\mathcal{X}_1) = 1$, we have that $\omega(y) \le K+1$ for $y \in \mathcal{X}$.

The case $\lambda_{y_1} = \gamma$ is similar.

(ii) Assume that $e_{y_1} < 0$.

(a) Assume that $\lambda_{y_1} = \gamma^{-1}$. Then $\mathcal{X}_2\mathcal{X}_1 = \gamma^{-1}\delta^{-1}\alpha\beta\dots$, and $h_1(\mathcal{X}_2\mathcal{X}_1) = 0$. Hence, if $\omega(y_0) \le K+1$ we must have $\omega(y) \le K+1$ for $y \in \mathcal{X}$.

The case $\lambda_{y_1} = \beta^{-1}$ is similar.

(b) Assume that $\lambda_{y_1} = \delta^{-1}$. Then $\mathcal{X}_2\mathcal{X}_1 = \delta^{-1}\alpha\beta\gamma^{-1}\dots$, and $h_k = \omega(y_0) - 1 \le K$. Hence, $h_k \le K-1$, which implies that $\omega(y_0) \le K$. Since $h_1(\mathcal{X}_2\mathcal{X}_1) = 1$, this implies the desired result.

The case $\lambda_{y_1} = \alpha^{-1}$ is similar to this one.

(B) We do not work out the case $\omega(y_0) > K+1$, since the proof is similar to the one just given.

LEMMA 4.4. *Let $y_0 = N_A + N_H + r_1$. Assume that $\lambda_{y_0} = \lambda_{y_0+1}$, $e_{y_0} = -e_{y_0+1}$, and $h_k \ne K$ or $K+1$. Then $\omega(y_0) \le K+1$ if and only if $\omega(y) \le K+1$ for all $y \in \mathcal{X}$.*

PROOF. The proof of this lemma is very similar to the proof of the preceding lemma so we do not give the details.

We now return to the proof of Proposition 4.1. Assume that Γ'_k, h_k, and $\mathcal{X}$ are as in (4.6). The proof is split into a number of cases. We assume throughout that $h_k \ne K$ or $K+1$.

(I) Assume that $H \ne \varnothing$. Then no further cancellations take place. In this case $||\Gamma_{k-1}|| = ||\Gamma_k||$. The result then follows from Lemma 4.3.

(II) Assume that $H = \varnothing$ and there are no further cancellations. There are two cases to consider. First suppose that

$$\mathcal{X} \subset Z_0^k = \{z \in Z^k : G_z = \varnothing\}.$$

Say

$$Z_0^{k-1} = \lambda_1^{e_1}\lambda_2^{e_2}\cdots\lambda_J^{e_J}$$

and

$$Z_0^k = \lambda_1^{e_1}\cdots\lambda_l^{e_l}\mathcal{X}_2\mathcal{X}_1\lambda_{l+1}^{e_{l+1}}\cdots\lambda_J^{e_J}.$$

By assumption $||\Gamma_{k-1}|| \le K+1$. Therefore, $\omega(\lambda_l) \le K+1$. The result then follows from Lemma 4.3.

Next assume that $\mathcal{X} \not\subset Z_k^0$. Then $||\Gamma_k|| = ||\Gamma_{k-1}|| \leq K+1$, and the result follows from Lemma 4.3.

(III) Assume that $H = \emptyset$ and there are cancellations. We show that everything is fine after one cancellation. Since there are a finite number of cancellations, this will imply the desired result. Now $\Gamma'_k = \Gamma_A \mathcal{X}_2 \mathcal{X}_1 \Gamma_B$. Assume that $\mathcal{X}_2\mathcal{X}_1 = \mathcal{X}_a\mathcal{X}_b\mathcal{X}_c$ where $\mathcal{X}_a$ and $\mathcal{X}_c$ have already cancelled. That is

$$\Gamma_A = \Gamma'_A \mathcal{X}_a^{-1} \quad \text{and} \quad \Gamma_B = \mathcal{X}_c^{-1}\Gamma'_B.$$

Then

$$\Gamma'_k = \Gamma'_A \mathcal{X}_b \Gamma'_B. \tag{4.8}$$

We assume that Γ'_k, as given in (4.8), has the desired properties; that is, it satisfies (a) and (b) of Proposition 4.1. We show that nothing goes wrong if there is one more cancellation. For now we assume that $\mathcal{X}_b \neq \emptyset$. There are many subcases to consider. We only give a detailed proof for a few of them.

(A) Suppose that $\Gamma'_A = \Gamma''_A \lambda^e$ and $\mathcal{X}_b = \lambda^{-e}\mathcal{X}'_b$. Then $\Gamma'_k = \Gamma''_A \mathcal{X}'_b \Gamma_B$.

(a) Suppose that $\lambda \in Z_0^{k-1}$. By assumption, $\omega(\lambda) \leq K+1$. But, by Lemma 4.2, $h_k \leq \omega(\lambda)$. Since $h_k \neq K$ or $K+1$ it follows that $h_k \leq K-1$. From Lemma 4.2 we conclude the $\omega(y) \leq K+1$ for all $y \in \mathcal{X}'_0$, and the result follows.

(b) Suppose that $\lambda \in Z_{k-1} \backslash Z_0^{k-1}$. The result then follows from Lemma 4.3, as before.

(c) Assume that $\lambda = \eta^{-1} \in H_{k-1}$. Then Γ'_k is of the form

$$\Gamma'_k = \underbrace{\bar{\Gamma}_A\, \lambda_1\, H_1\, \eta\, H_2\, \eta^{-1}}_{\Gamma''_A}\ \eta \mathcal{X}'_b\, \Gamma_B = \bar{\Gamma}_A\, \lambda_1\, H_1\, \eta\, H_2\, \mathcal{X}'_b\, \Gamma_B,$$

where $\psi(H_1) = \psi(H_2) = \mathcal{I}$ and $\lambda_1 = \max\{g_\eta, z(\eta)\}$. Here, as in (4.1),

$$g_\eta = \sup\{g \in G_\eta\} \quad \text{and} \quad z(\eta) = \{z \in Z\colon z < \eta, g_z = g_\eta\}.$$

It suffices to prove that if $y \in z(\eta) \cup \mathcal{X}'_b$, then $\omega(y) \leq K+1$ if and only if $\omega(\lambda_1) \leq K+1$. Since $\psi(H_1) = \psi(H_2) = \mathcal{I}$, we have that $\omega(\lambda_1) = \omega(\eta^{-1})$.

(c1) Assume that $\omega(\eta^{-1}) \leq K+1$. Then $\omega(\lambda_1) \leq K+1$, and therefore $\omega(y) \leq K+1$ for all $y \in \mathcal{X}'_b$ by Lemma 4.3. We must still worry if $y \in z(\eta)$. There are two subcases to consider. Assume that $\eta = \eta_1^{e_1}$.

(c1i) Assume that $e_1 > 0$. In the proof of Lemma 4.3 we showed that this implies that $\omega(\eta^{-1}) \leq K$, which implies that $\omega(\lambda_1) \leq K$. Since $\psi(H_1) = \mathcal{I}$, this implies that $\omega(\eta) \leq K+1$. Hence, $\omega(y) \leq K+1$ for $y \in z(\eta)$ because, in Γ_{k-1}, $\omega(\eta) \leq K+1$ if and only if $\omega(y) \leq K+1$ for $y \in z_\eta$.

(c1ii) Assume that $e_1 < 0$. Then $\omega(\lambda_1) \leq K+1$ and $\omega(\eta) \leq K+1$ implies that $\omega(y) \leq K+1$ for $y \in z(\eta)$, because this is true in Γ_{k-1}.

(c2) Assume that $\omega(\eta^{-1}) \leq K+1$. Then $\omega(\lambda_1) > K+1$ and $\omega(y) > K+1$ for all $y \in \mathcal{X}'_b$. The proof then proceeds as in (c1).

(d) Assume that $\lambda \in G$. This case is handled as above so we do not work out the details.

(B) Suppose that $\mathcal{X}_b = \mathcal{X}'_b\lambda^e$ and $\Gamma'_B = \lambda^{-e}\Gamma''_B$. The details of the proof in this case are similar to what was done for (A).

Finally, we must consider the case $\mathcal{X}_b = \emptyset$. Once again we do not give a detailed proof because the details are similar to the proofs given.

5. Completion. In this section we prove (1E.4b); that is,

PROPOSITION 5.1. *For each positive integer K there exists M such that if $k > M$, then $\|\Gamma_k\| > K$.*

As we pointed out in the remark following Proposition 4.1, this will complete the proof of the theorem.

In order to prove Proposition 5.1 we must recall where all of the algebraic objects came from. Hence, we must recall the notation and results in [**1** and **2**]. Because it would be very tedious to describe all of this material here, we assume that the reader is familiar with the notation and results in [**1** and **2**].

Recall Proposition 2.5 of [**2**]. This result states

PROPOSITION 5.2. *Given K there exists θ_K such that if $0 < \theta < \theta_K$, $0 \leq \varphi \leq 2\pi$, $d = (\varphi, \theta)$, and $U(d)(z) = B$ for some z, then $h(d) > K$.*

This will be the key ingredient in the proof of Proposition 5.1.

We must now introduce quite a bit of notation. We let A, B, C, X_1, X_2, X_3, W, $\bar{V}$, N_1, N, and $\mathcal{E}$ be as in [**2**]. Recall that

$$N_1 = \{(U_1, U_2) \colon |U_1| \leq W \text{ and } |U_2| \leq W\},$$
$$N = \{(U, V) \colon U \in N_1 \text{ and } \|V\| \leq \bar{V}\} \backslash (P_D \cup P_E),$$

and

$$\mathcal{E} = \{(U, V) \in \partial N \colon \|V\| < \bar{V}\}.$$

Let $I = [0, 1]$ be the unit interval.

DEFINITION. We say that $\Phi \colon I \times I \to N$ is an element of $\mathcal{S}$ if

(5.1)
(a) Φ is continuous, and for each t, the curve $\Phi(\cdot, t)$ is continuously differentiable,
(b) $\Phi(0, t) = (A, \mathcal{O})$ for all t,
(c) $\Phi(1, t) \in \mathcal{E}$ for all t,
(d) $\Phi(s, t) \notin \{(U, V) \colon F(U) = C\}$ for all (s, t),
(e) $\Phi(s, 0) \in \{(U, V) \colon U \in X_2\}$ for all s, and $\Phi(1, 0) \in \{(U, V) \colon U_2 = W\}$,
(f) $\Phi(s, 1) \in \{(U, V) \colon U \in X_2\}$ for all s, and $\Phi(1, 1) \in \{(U, V) \colon U_2 = -W\}$.

We note that elements of $\mathcal{S}$ arise quite naturally in the situations we are studying. If we let $g \in \mathcal{G}$ be as in [**2**], then $g \colon I \to Y$, which is also defined in [**2**]. For each $s \in I$, $g(s)$ corresponds to a trajectory $\gamma(s)(z)$ which lies in the unstable manifold of $(A, \mathcal{O})$. Moreover, $\gamma(s)(z)$ leaves N through $\mathcal{E}$. Hence, we may reparametrize $\gamma(s)(z)$, to say $\hat{\gamma}(s)(t)$, such that $0 \leq t \leq 1$, $\hat{\gamma}(s)(0) = (A, \mathcal{O})$, $\hat{\gamma}(s)(t) \in N$ for $0 \leq t \leq 1$, and $\hat{\gamma}(s)(1) \in \mathcal{E}$. Certainly, we may change the reparametrization to depend continuously on s. Hence, $\Phi(s, t) = \hat{\gamma}(s)(t)$ is in $\mathcal{S}$.

Now if $\Phi \in \mathcal{S}$, then $\Phi(1, t)$, $t \in [0, 1]$, defines a curve in $\mathcal{E}$. As in [**2**], we can assign to $\Phi(1, t)$ two algebraic objects, $\Gamma(\Phi)$ and $\Gamma^*(\Phi)$. (We always assume that a g-partition of Φ is given.) These are elements of F_4, the free group on the four elements $\{\alpha, \beta, \gamma, \delta\}$.

For each $t \in [0, 1]$, the curve $\Phi(s, t)$, $0 \leq s \leq 1$, winds around the sets P_D and P_E which were defined in §1C of this paper. As in that section we can define the winding number of that curve. We denote this winding number by $h(t)$.

DEFINITION. We say that $\Phi \in \mathcal{S}$ *crosses over* B *with order* K if there exists (s,t) such that $\Phi(s,t) \in \{(U,V)\colon U = B\}$ and $h(t) \le K$.

DEFINITION. For $Z \in F_4$ we say that $Z \in F_4'$ if there exists $\Phi \in \mathcal{S}$ with $\Gamma^*(\Phi) = Z$.

DEFINITION. If $Z \in F_4'$, then we say that Z *has order* K if for all $\Phi \in \mathcal{S}$ such that $\Gamma^*(\Phi) = Z$, Φ crosses over B with order K.

Let F_2 be the set of words on the two elements a, b. Let ψ be the homomorphism from F_4 onto F_2 generated by

$$\psi(\alpha) = \psi(\delta) = a \quad \text{and} \quad \psi(\beta) = \psi(\gamma) = d. \tag{5.2}$$

Let I be the identity element in F_2.

PROPOSITION 5.3. *Suppose that* Γ_A, Γ_B *and* Z *are elements of* F_4 *such that* $\psi(Z) = I$. *If* $\Gamma_A Z \Gamma_B \in F_4'$, *then* $\Gamma_A \Gamma_B \in F_4'$.

In order to prove this result we must introduce some notation and present some preliminary lemmas.

DEFINITION. Say that $\gamma\colon I \to N$ is an element of $\mathcal{H}$ if γ is continuously differentiable, $\gamma(0) = (A, \mathcal{O})$, and $\gamma(1) \in \mathcal{E}$.

If $\gamma \in \mathcal{H}$, then $\gamma(s)$ winds around P_D and P_E. We now show that γ generates an element of F_2, the fundamental group of N. We denote the element by $\theta(\gamma)$, and define $\theta(\gamma)$ explicitly as follows. Let

$$Q_D^- = \{(U,V) \in N\colon U_1 = D_1,\ V_1 < 0\}$$

and

$$Q_E^+ = \{(U,V) \in N\colon U_1 = E_1,\ V_1 > 0\}.$$

We assume, for convenience, that $\gamma(s)$ intersects Q_D^- and Q_D^+ only a finite number of times. This will be true in the situations we are interested in. Choose $\eta_1 < \eta_2 < \cdots < \eta_K$ such that $\gamma(\eta) \in Q_D^+ \cup Q_E^+$ if and only if $\eta = \eta_k$ for some k. Assume that $\gamma(s) = (U(s), V(s))$. If

(a) $\gamma(\eta_k) \in Q_D^-$ and $U'(\eta_k) < 0$, let $\lambda_k = b$ and $e_k = +1$,
(b) $\gamma(\eta_k) \in Q_D^-$ and $U'(\eta_k) > 0$, let $\lambda_k = b$ and $e_k = -1$,
(c) $\gamma(\eta_k) \in Q_E^+$ and $U'(\eta_k) < 0$, let $\lambda_k = a$ and $e_k = -1$,
(d) $\gamma(\eta_k) \in Q_E^+$ and $U'(\eta_k) > 0$, let $\lambda_k = a$ and $e_k = +1$.

Let

$$\theta^*(\gamma) = \lambda_1^{e_1} \lambda_2^{e_2} \cdots \lambda_K^{e_K}$$

and $\theta(\gamma)$ equal $\theta^*(\gamma)$ with all cancellations.

Note that if $d \in \mathcal{D}$ then $\gamma(d) \in \mathcal{H}$ once we reparametrize $\gamma(d)(z)$ to say $\hat{\gamma}(d)(s)$ where $\hat{\gamma}(d)(0) = \gamma(d)(-\infty) = (A, \mathcal{O})$ and $\hat{\gamma}(d)(1) \in \mathcal{E}$.

LEMMA 5.4. *If* $d \in \mathcal{D}$ *and* $\theta^*(\hat{\gamma}(d)) = \lambda_1^{e_1} \cdots \lambda_K^{e_K}$, *then, for each* i, $e_i > 0$.

PROOF. Since $U_1' = V_1$, it follows that if $\gamma(d)(z_0) \in Q_D^-$ then $U_1'(z_0) = V_1(z_0) < 0$. If $\gamma(d)(z_1) \in Q_D^+$, then $U_1'(z_1) = V_1(z_1) > 0$.

We immediately have

COROLLARY 5.5. *If* $d \in \mathcal{D}$, *then* $\theta^*(\hat{\gamma}(d)) = \theta(\hat{\gamma}(d))$.

That is, no cancellations take place in $\theta^*(\hat{\gamma}(d))$.

Note that if $\Phi \in \mathcal{S}$ then for each t, $0 \le t \le 1$, $\Phi(\cdot, t) \in \mathcal{H}$. Because of Corollary 5.5 we add the following condition to elements of $\mathcal{S}$:

(5.1g) If $\Phi \in \mathcal{S}$, then $\theta^*(\Phi(\cdot,t)) = \theta(\Phi(\cdot,t))$ for each $t \in I$.

We also assume that each $\gamma \in \mathcal{H}$ satisfies

$$\theta^*(\gamma) = \theta(\gamma). \tag{5.3}$$

Because $\theta(\gamma)$ is really just the element of the fundamental group of N generated by $\gamma(s)$ we have

LEMMA 5.6. *Assume that* $\gamma_1 \in \mathcal{H}$, $\gamma_2 \in \mathcal{H}$, *and* $\theta(\gamma_1) = \theta(\gamma_2)$ (*and therefore* $\theta^*(\gamma_1) = \theta^*(\gamma_2)$). *Then* γ_1 *is homotopic to* γ_2. *That is, there exists a continuous map* $h: I \times I \to N$ *such that*

(a) $h(\cdot, t) \in \mathcal{H}$ *for each* t,
(b) $h(s, 0) = \gamma_1(s)$ *for each* s,
(c) $h(s, 1) = \gamma_2(s)$ *for each* s,
(d) $\theta(h(\cdot, t)) = \theta(\gamma_1)$ *for each* t.

Now fix $\Phi \in \mathcal{S}$. We shall show that Φ induces a map $\Lambda(\Phi): [0,1] \to F_4$. Fix $t_0 \in [0,1]$. Then $\bigcup_{t \in [0,t_0]} \Phi(1,t)$ defines a curve in $\mathcal{E}$. As before, we can define elements $\Gamma^*_{t_0}(\Phi)$ and $\Gamma_{t_0}(\Phi)$ of F_4. The definition of $\Gamma^*_{t_0}(\Phi)$ is precisely as of $\Gamma^*(\Phi)$. Let $\Lambda(\Phi)(t_0) = \Gamma_{t_0}(\Phi)$. For each t_0, $\Phi(\cdot, t_0) \in \mathcal{H}$. A key result is

LEMMA 5.7. *Assume that* $\Phi \in \mathcal{S}$. *Then for each* $t_0 \in [0,1]$,

$$\psi(\Lambda(\Phi)(t_0)) = \theta(\Phi(\cdot, t_0)). \tag{5.4}$$

PROOF. Note that as $t \in [0,1]$ changes, then the curve $\Phi(1,t) \in \mathcal{E}$ changes, and the one parameter family of curves $\Phi(\cdot, t)$ changes. Hence, $\Lambda(\Phi)(t)$ and $\theta(\Phi)(t)$ change. We must show that they change according to the relation (5.4). Note that (5.4) certainly holds when $t_0 = 0$. In fact, $\psi(\Lambda(\Phi))(0) = \theta(\Phi(\cdot, 0)) = \mathcal{I}$, the identity element of F_2. This is because of (5.1e).

Let us now consider for which values of t_0 it is possible for $\Lambda(\Phi)(t_0)$ to change. From the definitions (see Table I of [**1**]) we have that $\Lambda(\Phi)(t)$ can only change at $t = t_0$, if $\Phi(1, t_0) \in l^+_\alpha \cup l^-_\beta \cup l^-_\gamma \cup l^+_\delta$ where l^+_α, l^-_β, l^-_γ, and l^+_δ were defined in [**1**]. That is,

$$\begin{aligned}
l^+_\alpha &= \{(U,V) \in \partial N : U_1 = E_1,\ U_2 = W,\ V_1 > 0\},\\
l^-_\beta &= \{(U,V) \in \partial N : U_1 = D_1,\ U_2 = W,\ V_1 > 0\},\\
l^-_\gamma &= \{(U,V) \in \partial N : U_1 = D_1,\ U_2 = -W,\ V < 0\},\\
l^+_\delta &= \{(U,V) \in \partial N : U_1 = E_1,\ U_2 = -W,\ V_1 > 0\}.
\end{aligned}$$

The rules for how $\Lambda(\Phi)(t)$ changes at t_0 are given in Table I of [**1**].

Let us now consider for which values of t_0 it is possible for $\Theta(\Phi(\cdot, t_0))$ to change. Clearly, $\Theta(\Phi(\cdot, t_0))$ changes at those values of t_0 at which the curves $\Phi(\cdot, t)$ cross Q^-_D or Q^+_E. There is only one way, as we now show, for $\Phi(\cdot, t)$ to cross Q^-_D or Q^+_E at $t = t_0$. This is if $\Phi(1, t_0) \in \partial N \cap (Q^-_D \cup Q^+_E)$. Certainly this is one possible way for

$\Phi(\cdot,t)$ to cross Q_D^- and Q_E^+ at $t=t_0$. Another possibility, which we now rule out, is for $\Phi(s,t_0)$ to be tangent to Q_D^- or Q_E^+ for some $s\in(0,1)$. This is impossible, however, because of our assumption (5.1g). If $\Phi(s,t_0)$ were tangent to Q_D^- or Q_E^+ for some $s\in(0,1)$ we would not have that $\theta^*(\Phi(0,t_0))=\theta(\Phi(0,t))$.

We have now shown that $\Theta(\Phi(\cdot,t))$ can only change at $t=t_0$ if $\Phi(1,t_0)\in \partial N\cap(Q_D^-\cup Q_E^+)$. However,

$$\partial N\cap(Q_D^-\cup Q_E^+)=l_\alpha^+\cup l_\beta^-\cup l_\gamma^-\cup l_\delta^+.$$

So we have that $\Lambda(\Phi)(t)$ and $\Theta(\Phi(\cdot,t))$ can only change at the same values of t. It remains to show that they change according to (5.4). This, however, follows from the definitions. (In fact, the definitions were chosen precisely so that (5.4) would be valid.)

We are now ready to complete the

PROOF OF PROPOSITION 5.3. Suppose that

$$\Gamma_A=\lambda_1^{e_1}\cdots\lambda_J^{e_J},\quad Z=\lambda_{J+1}^{e_{J+1}}\cdots\lambda_K^{e_K},\quad \Gamma_B=\lambda_{K+1}^{e_{K+1}}\cdots\lambda_L^{e_L},$$

where each $\lambda_i\in\{\alpha,\beta,\gamma,\delta\}$ and $e_i\in\{-1,1\}$. By assumption, $\Gamma_AZ\Gamma_B\in F_4'$. Choose $\Phi\in S$ so that $\Gamma^*(\Phi)=\Gamma_AZ\Gamma_B$. Recall that to define $\Gamma^*(\Phi)$ we need to start with a g-partition $\eta^*=\{\eta_0,\eta_1,\dots,\eta_L\}$. Here we are using the language and notation of [**2**]. Then for $i=1,\dots,L$, $\lambda_i=\lambda(\eta_i)$ and $e_i=e(\eta_i)$. Fix $\sigma_1\in(\eta_J,\eta_{J+1})$ and $\sigma_2\in(\eta_K,\eta_{K+1})$. Then $\Lambda(\Phi)(\sigma_1)=\Gamma_A$ and $\Lambda(\Phi)(\sigma_2)=\Gamma_AZ$. Because $\psi(Z)=I$, we conclude that

$$\psi(\Lambda(\Phi)(\sigma_1))=\psi(\Lambda(\Phi)(\sigma_2)).$$

From Lemma 5.7 it follows that

$$\Theta(\Phi(\cdot,\sigma_1))=\Theta(\Phi(\cdot,\sigma_2)).$$

From Lemma 5.6 we conclude that there exists a continuous map $h\colon I\times I\to N$ such that

(a) $h(\cdot,t)\in\mathcal{H}$ for each t,
(b) $h(s,0)=\Phi(s,\sigma_1)$ for each s,
(c) $h(s,1)=\Phi(s,\sigma_2)$ for each s,
(d) $\Theta(h(\cdot,t))=\theta(\Phi(\cdot,\sigma_1))$ for each t.

We are trying to prove that $\Gamma_A\Gamma_B\in F_4'$. That is, we must prove that there exists $\Phi'\in S$ such that $\Gamma^*(\Phi')=\Gamma_A\Gamma_B$. The following Φ' does the job. Let

$$\Phi'(s,t)=\begin{cases}\Phi(s,t) & \text{for } s\in[0,1],\ 0\le t\le\sigma_1,\\ h\left(s,\dfrac{t-\sigma_1}{\sigma_2-\sigma_1}\right) & \text{for } s\in[0,1],\ \sigma_1\le t\le\sigma_2,\\ \Phi(s,t) & \text{for } s\in[0,1],\ \sigma_2\le t\le 1.\end{cases}$$

It is not hard to prove that $\Gamma^*(\Phi')=\Gamma_A\Gamma_B$ so we do not give the details.

From now on we assume that the $\{I_k\}$ are as in [**2**]. To each I_k we fix a g-partition η_k and let $\Gamma_k'=\Gamma^*(I_k,\eta^*)$ where $\Gamma^*(I_k,\eta^*)$ is as in [**2**]. Let Γ_k equal Γ_k' with all possible cancellations. In [**2**] we proved that the $\{\Gamma_k\}$ satisfy (a) and (c) of Proposition 1E.1. Of course, here we wish to prove that the $\{\Gamma_k\}$ satisfy (b) of Proposition 1E.1. Note that, in this paper we have shown that for each k, there exists an A^*-decomposition $\{Z_k,G_k,H_k,\Phi_k\}$ of Γ_k. Let Z_0^k be as in (2.5b). That is, $Z_0^k=\{z\in Z_k\colon G_z=\varnothing\}$. Let Γ_0^k be as in (2.6).

LEMMA 5.7. *For each k, $\Gamma'_k \in F'_4$.*

PROOF. Note that each $I_k \in \mathcal{G}$. As we pointed out after the definition of $\mathcal{S}$, each element of $\mathcal{G}$ gives rise to an element of $\mathcal{S}$. That is, we reparametrize each trajectory $\gamma(\Phi(s))(z)$ to say $\hat{\gamma}(\Phi(s))(t)$ so that $\hat{\gamma}(\Phi(s))(0) = \gamma(\Phi(s))(-\infty) = (A, \mathcal{O})$ and $\hat{\gamma}(\Phi(s))(1) \in \mathcal{E}$ for each $s \in [0,1]$. Let $\Phi(s,t) = \hat{\gamma}(\Phi(s))(t)$. From the definitions it is clear that $\Gamma'_k = \Gamma^*(\hat{\gamma}(\Phi(s)))$ once we choose the appropriate g-partition.

LEMMA 5.8. *For each k, $\Gamma_k \in F'_4$.*

PROOF. Note that Γ_k is obtained from Γ'_k by a finite number of cancellations. After each cancellation we apply Proposition 5.3 to conclude that the element of F_4 obtained by the cancellation is still in F'_4.

LEMMA 5.9. *For each k, $\Gamma_0^k \in F'_4$.*

PROOF. Assume that $\Gamma_0^k = \lambda_1^{e_1}\lambda_2^{e_2}\cdots\lambda_L^{e_L}$ where each $\lambda_i \in \{\alpha,\beta,\gamma,\delta\}$ and $e_i \in \{-1,1\}$. Then there exists $Z_0, Z_1, \ldots, Z_L$ such that for each i, $Z_i \in F_4$, $\psi(Z_i) = I$, and

$$\Gamma_k = Z_0\lambda_1^{e_1}Z_1\lambda_2^{e_2}\cdots\lambda_L^{e_L}Z_L.$$

Hence, Γ_0^k is obtained from Γ_k by removing, one at a time, the Z_i. We apply Proposition 5.3 to conclude that each time we remove Z_i, the resulting word is in F'_4.

LEMMA 5.10. *Assume that $\Gamma_A\Gamma_B \in F'_4$ and $\Gamma_A g^e Z g^{-e}\Gamma_B \in F'_4$ where $g \in \{\alpha,\beta,\gamma,\delta\}$, $e \in \{-1,1\}$, $\psi(Z) = I$, and Z may be empty. Assume that for $z \in Z$, $\omega(z) \le K$ if and only if $\omega(g) \le K$. Finally, assume that $\Gamma_A\Gamma_B$ has order K. Then $\Gamma_A g Z g^{-1}\Gamma_B$ has order K.*

PROOF. Suppose, for the sake of a contradiction, that $\Gamma_A g Z g^{-1}\Gamma_B$ does not have order K. Then there exists $\Phi \in \mathcal{S}$ such that $\Gamma^*(\Phi) = \Gamma_A g Z g^{-1}\Gamma_B$, and if $\Phi(s,t) \in \{(U,V)\colon U = B\}$, then $h(t) > K$. From Φ we will construct a map $\Phi' \in \mathcal{S}$ such that $\Gamma(\Phi') = \Gamma_A\Gamma_B$, and if $\Phi'(s,t) \in \{(U,V)\colon U = B\}$, then $h(t) > K$. This will contradict the assumption that $\Gamma_A\Gamma_B$ has order K. The construction of Φ' will be similar to the construction in the proof of Proposition 5.3.

Suppose that

$$\Gamma_A = \lambda_1^{e_1}\cdots\lambda_{J-1}^{e_{J-1}}, \qquad g^e = \lambda_J^{e_J},$$
$$Z = \lambda_{J+1}^{e_{J+1}}\cdots\lambda_{L-1}^{e_{L-1}}, \quad g^{-e} = \lambda_L^{e_L}, \quad \Gamma_B = \lambda_{L+1}^{e_{L+1}}\cdots\lambda_N^{e_N}.$$

Now, $\Gamma^*(\Phi) = \Gamma_A g^e Z g^{-e}\Gamma_B$. In order to define $\Gamma^*(\Phi)$ we must start with a g-partition $\eta^* = \{\eta_0, \ldots, \eta_N\}$. For $i = 1, \ldots, N$, $\lambda_i = \lambda(\eta_i)$ and $e_i = e_i(\eta_i)$ where we are using the notation in [**2**]. Fix $\sigma_1 \in \{\eta_{J-1}, \eta_J\}$ and $\sigma_2 \in \{\eta_L, \eta_{L+1}\}$. Then

$$\Lambda(\Phi)(\sigma_1) = \Gamma_A \quad\text{and}\quad \Lambda(\Phi)(\sigma_2) = \Gamma_A g^e Z g^{-e}.$$

Because $\psi(Z) = I$, we conclude that

$$\psi(\Lambda(\Phi)(\sigma_1)) = \psi(\Lambda(\Phi)(\sigma_2)).$$

From Lemma 5.6 it follows that

$$\Theta(\Phi(\cdot,\sigma_1)) = \Theta(\Phi(\cdot,\sigma_2)).$$

We now apply Lemma 5.5 to conclude that there exists a continuous map $g\colon I\times I \to N$ such that

$$(5.5)\qquad \begin{array}{ll} \text{(a)} & g(\cdot,t) \in \mathcal{H} \text{ for each } t, \\ \text{(b)} & g(s,0) = \Phi(s,\sigma_1) \text{ for each } s, \\ \text{(c)} & g(s,1) = \Phi(s,\sigma_2) \text{ for each } s, \\ \text{(d)} & \Theta(h(\cdot,t)) = \Theta(\Phi(\cdot,\sigma_1)) \text{ for each } t. \end{array}$$

We then let

$$(5.6)\qquad \Phi'(s,t) = \begin{cases} \Phi(s,t) & \text{for } s\in[0,1],\ 0\le t<\sigma_1, \\ g\left(s, \dfrac{t-\sigma_1}{\sigma_2-\sigma_1}\right) & \text{for } s\in[0,1],\ \sigma_1\le t\le\sigma_2, \\ \Phi(s,t) & \text{for } s\in[0,1],\ \sigma_2\le t\le 1. \end{cases}$$

Certainly $\Phi' \in \mathcal{S}$, and $\Gamma^*(\Phi') = \Gamma_A\Gamma_B$. Moreover, if $t \notin [\sigma_1,\sigma_2]$, then $\Phi'(s,t) = \Phi(s,t)$. Hence, if $\Phi'(s,t) \in \{(U,V)\colon U = B\}$ and $t \notin [\sigma_1,\sigma_2]$, then $h(t) > K$. We claim that $g(s,t)$ can be chosen so that the same is true for $t \in [\sigma_1,\sigma_2]$. If $\omega(g) \le K$ then $\omega(z) \le K$ for all $z \in Z$. From our assumption on Φ this implies that there does not exist $t \in [\sigma_1,\sigma_2]$ such that $\Phi(s,t) \in \{(U,V)\colon U = B\}$ for some s. Hence, there is no problem in our choice of $g(s,t)$. So we assume that $\omega(g) > K$, and therefore $\omega(z) > K$ for all $z \in Z$.

For a given (s,t) let $h(s,t)$ be the winding number of the curve $\Phi(\sigma,t)$, $0 \le \sigma \le s$. Because $\omega(z) > K$ for all $z \in Z$, there exists a map $s = \varphi(t)$, $\sigma_1 \le t \le \sigma_2$, such that for all $t \in [\sigma_1,\sigma_2]$,

(a) $h(\varphi(t),t) > K$,

(b) $\Theta(\Phi(\varphi(t),t)) = \Theta(\Phi(\varphi(\sigma_1,\sigma_1)))$.

Recall that if $\Phi(s,t) \in \{(U,V)\colon U = B\}$, then $h(t) > K$. Assume that

$$\Phi'(s,t) = \Phi(s,t) \quad \text{for } \sigma_1 \le t \le \sigma_2,\ 0 \le s \le \varphi(t).$$

That is, we let, for $0 \le t \le 1$, $0 \le s \le \varphi((\sigma_2-\sigma_1)t+\sigma_1)$,

$$g(s,t) = \Phi(s,(\sigma_2-\sigma_1)t+\sigma_1).$$

Then for $0 \le t \le 1$, $(\sigma_2-\sigma_1)t+\sigma_1 < s \le 1$, we let $g(s,t)$ be arbitrary so that (5.5) is satisfied. One then checks that with this choice of $g(s,t)$, $\Phi'(s,t)$, as defined by (5.6), has the desired properties.

COMPLETION OF THE PROOF OF PROPOSITION 5.1. Let K be any positive integer. By Proposition 5.2 there exists θ_K such that if $0 < \theta < \theta_K$, $0 \le \varphi \le 2\pi$, $d = (\varphi,\theta)$, and $U(d)(z) = B$ for some z, then $h(d) > K$. Let I_k be the elements of $\mathcal{G}$ which generate the Γ_k and Γ_k^*. From the definitions of the I_k (see [**2**, (4A.1c)]) there exists M such that if $k > M$ and $I_k(s) = (\varphi(s),\theta(s))$, then $\theta(s) < \theta_K$ for each $s \in [0,1]$. We claim that if $k > M$, then $||\Gamma_k|| > K$.

Assume that $k > M$. We first show that Γ_k' does not have order K. This is because, as in the proof of Lemma 5.7, $I_k \in \mathcal{G}$ gives rise to an element $\Phi(s,t) \in \mathcal{S}$. Because $k > M$, $\Phi(s,t)$ has the property that if $\Phi(s,t) \in \{(U,V)\colon U = B\}$, then $h(t) > K$. Hence, $\Gamma^*(\Phi) = \Gamma'$ and Φ does not cross over B with order K. A finite number of applications of Lemma 5.10, with $Z = \varnothing$, implies that Γ_k does not have order K. We now apply Lemma 5.10 a finite number of times, again, and use Proposition 4.1b to conclude that Γ_0^k does not have order K. (Here, we assume for the sake of a contradiction, that there does not exist j such that $h_j = K$ or $K+1$.)

By Lemma 5.9, there exists $\Phi \in \mathcal{S}$ such that $\Gamma^*(\Phi) = \Gamma_0^k$. Moreover, we may choose Φ so that $\Phi(s_0 t_0) \in \{(U,V): U = B\}$ for some (s_0, t_0) and $h(t_0) > K$. In order to complete the proof we must apply Proposition 2.6 of [**2**].

To state this result we must present some notation.

Note that $\Phi(1,t)$, $0 \leq t \leq 1$, defines a curve in $\mathcal{E}$. This curve generates the elements $\Gamma^*(\Phi)$ and $\Gamma(\Phi)$. To define $\Gamma^*(\Phi)$ we must define a g-partition $\eta^* = \{\eta_1, \ldots, \eta_L\}$ (see [**1**]). We then define, for each j, $\lambda(\eta_j) \in \{\alpha, \beta, \gamma, \delta\}$ and $e(\eta_j) \in \{-1, 1\}$. Note that from the definition of a g-partition, we have $0 = \eta_1 < \cdots < \eta_L = 1$. Define the map $\Lambda: I \to F_4$ as follows. Suppose that $\eta_j \leq s < \eta_{j+1}$. Let $\lambda_i = \lambda(\eta_i)$ and $e_i = e(\eta_i)$. Then let

$$\Lambda(s) = \prod_{i=1}^{j} \lambda_i^{e_i} = \lambda_1^{e_1} \cdots \lambda_j^{e_j}.$$

Define $\Lambda_1: I \to Z^+$, where Z^+ is the set of nonnegative integers, by

$$\Lambda_1(x) = [\omega \circ \Lambda](s).$$

The map ω was defined in (1E.2). Then in [**2**, Proposition 2.6], we prove

PROPOSITION 5.10. *Let $\eta^* = \{\eta_1, \ldots, \eta_L\}$ be a g-partition of the curve $\Phi(1,t)$, $0 \leq t \leq 1$. Assume that $\eta_j \leq t < \eta_{j+1}$. Then either*

$$h(t) = \Lambda_1(\eta_j) \quad \textit{or} \quad h(t) = \Lambda_1(\eta_{j+1}). \tag{5.7}$$

With Proposition 5.10 the proof of Proposition 5.1 now follows easily. We know that $h(t_0) > K$ for some t_0. From (5.7) we conclude that if $\eta_j \leq t_0 < \eta_{j+1}$, then either $\Lambda_1(\eta_j) > K$ or $\Lambda_1(\eta_{j+1}) > K$. However,

$$||\Gamma_k|| = h_1(\Gamma_0^k) = \sup_{1 \leq i \leq L} \Lambda_1(\eta_j) \geq \max\{\Lambda_1(\eta_j), \Lambda_1(\eta_{j+1})\} > K,$$

which completes the proof.

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